Title: *Multi-Site Damage Localization Using Least-Squares Optimization with Low Rank SVD Updates*

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ABSTRACT

An improvement to an optimal solution method, previously described by the authors, is proposed for determining multiple damaged locations within a perturbed system utilizing a nonlinear quasi-second order search algorithm imbedded within a combinatorial search. The proposed method utilizes low-rank updates of the singular value decomposition (SVD) to facilitate efficient function evaluations.

Building on previous work by the authors, this algorithm is extended to efficiently find multiple damage locations in a system. In the original algorithm, a search of multiple damage sites requires a SVD for every possible combination of damage locations. To alleviate the computational burden associated with this, a low-rank SVD update method was devised such that special SVD matrices are pre-calculated and used to construct the aggregate SVD used by the least-squares optimization. Using this method, the number of times the SVD needs to be computed is reduced to a value on the order of the number of possible damage locations. This method results in significantly less SVD operations overall, making the ability to localize simultaneous damage in multiple locations computationally feasible. This technique is presented in the context of simulating damage in a damped, vibrating system at two arbitrary locations.

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INTRODUCTION

This document presents a method to localize damage at multiple locations in a structure based on the method presented by D. L. Parker et al in [4]. Much like Parker's method, this new method relies on the concept that given an appropriate model of the inherent dynamics of a structure and the measured frequency responses of the structure at both an undamaged state and a damaged state, an estimation of the change in model parameters can be performed to adjust the model to fit the measured frequency response of the damaged state. However, the method proposed in [4] is further extended to efficiently account for and localize the damage at multiple locations.

Like the method in [4], the method described in this paper performs a localization operation by utilizing the Levenberg-Marquardt optimization [6] method and the Sherman-Morrison-Woodbury formula [2] for matrix inversion. In addition, a method for computing low-rank updates to the singular value decomposition [5] is proposed for handling coupled degrees of freedom between multiple damaged locations.

BACKGROUND

In the previous method, estimating the damage at multiple locations requires the computation of an SVD with an order equivalent to the order of the combined damage location degrees of freedom for all iterations of the non-linear fit operation. This paper addresses this drawback by proposing a method that provides for pre-computation of the low-rank SVDs of each damage location's sensitivity matrices and combines them into the appropriate higher-rank SVD needed for all iterations of the non-linear fit operation with minimal computations. This ultimately results in a greatly decreased search time when localizing and estimating damage at multiple sites.

Consider the perturbed transfer function of a canonical second-order form of a damped vibrating system with N_{dof} degrees-of-freedom (DOF).

$$T(s) + \Delta T(s) = s^2 C_a (s^2 M + sG + K + X)^{-1} F \,. \tag{1}$$

Consider that X represents multiple $N_{\Delta K}$-rank ΔK perturbations in the stiffness matrix K such that

$$\begin{aligned} &rank(X) << rank(K) \\ &X = \alpha_{\Delta K_1} \Delta K_1 + \alpha_{\Delta K_2} \Delta K_2 + \ldots + \alpha_{\Delta K_n} \Delta K_n \,, \\ &\Delta K_i = U_{\Delta K_i} \Sigma_{\Delta K_i} V_{\Delta K_i}^T \end{aligned} \tag{2}$$

where n is the number of locations at which damage is estimated. In other words, the matrix X is the sum of multiple perturbations, each corresponding to a damage location in the finite element model.

In the single-site damage localization technique described in [4], the Sherman-Morrison-Woodbury formula for matrix inversion was utilized to calculate $\Delta T(s)$ for a single damage location [2]. Rewriting gives us the following equation

$$\begin{aligned}\Delta T(s) = &-s^2 C_a (s^2 M + sG + K)^{-1} U_X \bullet \\ &\left[\Sigma_X^{-1} + V_X^T (s^2 M + sG + K)^{-1} U_X\right]^{-1} \bullet \\ &V_X^T (s^2 M + sG + K)^{-1} F\end{aligned}, \tag{3}$$

where U_X, V_X, and Σ_X are the singular value decomposition matrices satisfying the following equality

$$X = U_X \Sigma_X V_X^T. \tag{4}$$

Extending this to the multi-site case described by (2), results in

$$U_X \Sigma_X V_X^T = \sum_{i=1}^{n} \alpha_{\Delta K_i} U_{\Delta K_i} \Sigma_{\Delta K_i} V_{\Delta K_i}^T, \tag{5}$$

where $\alpha_{\Delta K_i} \in \Re$, $U_X, V_X \in \Re^{N_{dof} \times N_X}$, $\Sigma_X \in \Re^{N_X \times N_X}$, $U_{\Delta K_1}, V_{\Delta K_1} \in \Re^{N_{dof} \times N_{\Delta K_i}}$, and $\Sigma_{\Delta K_1} \in \Re^{N_{\Delta K_i} \times N_{\Delta K_i}}$. While this method can work with *n* number of damage locations, in this example damage is estimated at only two possible locations. This leads to the specific scenario where

$$X = \alpha_{\Delta K_1} \Delta K_1 + \alpha_{\Delta K_2} \Delta K_2. \tag{6}$$

Here, we can express the singular value decomposition of X as

$$X = \begin{bmatrix} U_{\Delta K_1} & \alpha_{\Delta K_2} L_{\Delta K_2} \end{bmatrix} \begin{bmatrix} \alpha_{\Delta K_1} \Sigma_{\Delta K_1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} V_{\Delta K_1} & R_{\Delta K_2} \end{bmatrix}^T, \tag{7}$$

where $\Delta K_1 \in \Re^{p \times q}$ have rank r and thin-SVD $\alpha_{\Delta K_1} U_{\Delta K_1} \Sigma_{\Delta K_1} V_{\Delta K_1}^T = \Delta K_1$, with $\Sigma_{\Delta K_1} \in \Re^{r \times r}$, and $\Delta K_2 \in \Re^{p \times q}$ have rank c and can be decomposed into matrices $L_{\Delta K_2} \in \Re^{p \times c}$ and $R_{\Delta K_2} \in \Re^{c \times q}$ such that

$$\alpha_{\Delta K_2} \Delta K_2 = \alpha_{\Delta K_2} L_{\Delta K_2} R_{\Delta K_2}, \tag{8}$$

where $L_{\Delta K_i} = U_{\Delta K_i} \Sigma_{\Delta K_i}$, $R_{\Delta K_2} = V_{\Delta K_i}^T$, and

$$rank(\Delta K_1 + \Delta K_2) \le r + c < \min(p, q). \tag{9}$$

Let $P_{\Delta K_1 \perp \Delta K_2}$ be an orthogonal basis of the column-space of $(I - U_{\Delta K_1} U_{\Delta K_1}^T)\alpha_{\Delta K_2} L_{\Delta K_2}$ and set $\Psi_{L_{\Delta K_1 \perp \Delta K_2}} = P_{\Delta K_1 \perp \Delta K_2}^T (I - U_{\Delta K_1} U_{\Delta K_1}^T)\alpha_{\Delta K_2} L_{\Delta K_1}$. Note that $cols(P_{\Delta K_1 \perp \Delta K_2}) = rows(\Psi_{\Delta K_1 \perp \Delta K_2}) = rank((I - U_{\Delta K_1} U_{\Delta K_1}^T)L_{\Delta K_2}) \le c$, and may be equal to zero in the case that ΔK_1 and ΔK_2 are orthogonal to each other. This relationship is summarized as

$$\begin{bmatrix} U_{\Delta K_1} & \alpha_{\Delta K_2} L_{\Delta K_2} \end{bmatrix} = \begin{bmatrix} U_{\Delta K_1} & P_{\Delta K_1 \perp \Delta K_2} \end{bmatrix} \begin{bmatrix} I & U_{\Delta K_1}^T \alpha_{\Delta K_2} L_{\Delta K_2} \\ 0 & \Psi_{\Delta K_1 \perp \Delta K_2} \end{bmatrix}. \tag{10}$$

Similarly, let $Q_{\Delta K_1 \perp \Delta K_2}$ be an orthogonal basis of the column-space of $(I - V_{\Delta K_1} V_{\Delta K_1}^T)R_{\Delta K_2}$, and set $\Theta_{\Delta K_1 \perp \Delta K_2} = Q_{\Delta K_1 \perp \Delta K_2}^T (I - V_{\Delta K_1} V_{\Delta K_1}^T)R_{\Delta K_2}$. Knowing this, we can perform the following substitutions

$$\Delta K_1 + \Delta K_2 = \begin{bmatrix} U_{\Delta K_1} & P_{\Delta K_1 \perp \Delta K_2} \end{bmatrix} \Lambda \begin{bmatrix} V_{\Delta K_1} & Q_{\Delta K_1 \perp \Delta K_2} \end{bmatrix}^T. \tag{11}$$

This produces a product of two orthonormal matrices, where

$$\begin{aligned} \Lambda &= \begin{bmatrix} I & U_{\Delta K_1}^T \alpha_{\Delta K_2} L_{\Delta K_2} \\ 0 & \Psi_{\Delta K_1 \perp \Delta K_2} \end{bmatrix} \begin{bmatrix} \alpha_{\Delta K_1} \Sigma_{\Delta K_1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & V_{\Delta K_1}^T R_{\Delta K_2} \\ 0 & \Theta_{\Delta K_1 \perp \Delta K_2} \end{bmatrix}^T \\ &= \begin{bmatrix} \alpha_{\Delta K_1} \Sigma_{\Delta K_1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} U_{\Delta K_1}^T \alpha_{\Delta K_2} L_{\Delta K_2} \\ \Psi_{\Delta K_1 \perp \Delta K_2} \end{bmatrix} \begin{bmatrix} V_{\Delta K_1}^T R_{\Delta K_2} \\ \Theta_{\Delta K_1 \perp \Delta K_2} \end{bmatrix}^T \end{aligned}. \tag{12}$$

Performing a singular value decomposition on Λ gives us the following

$$\Lambda = U_\Lambda \Sigma_\Lambda V_\Lambda^T. \tag{13}$$

where U_Λ and V_Λ are the rotations applied to the extended subspaces $\begin{bmatrix} U_{\Delta K_1} & P_{\Delta K_1 \perp \Delta K_2} \end{bmatrix}$ and $\begin{bmatrix} V_{\Delta K_1} & Q_{\Delta K_1 \perp \Delta K_2} \end{bmatrix}$ such that

$$X = \Delta K_1 + \Delta K_2 = \left(\begin{bmatrix} U_{\Delta K_1} & P_{\Delta K_1 \perp \Delta K_2} \end{bmatrix} U_\Lambda \right) \Sigma_\Lambda \left(\begin{bmatrix} V_{\Delta K_1} & Q_{\Delta K_1 \perp \Delta K_2} \end{bmatrix} V_\Lambda \right)^T \tag{14}$$

From (4), we see that

$$U_X = \left(\begin{bmatrix} U_{\Delta K_1} & P_{\Delta K_1 \perp \Delta K_2} \end{bmatrix} U_\Lambda \right), \tag{15}$$

$$\Sigma_X = \Sigma_\Lambda, \tag{16}$$

$$V_X = \left(\begin{bmatrix} V_{\Delta K_1} & Q_{\Delta K_1 \perp \Delta K_2} \end{bmatrix} V_\Lambda\right). \tag{17}$$

Performing this analysis in a recursive fashion, this can be extended to n monitored damage locations. Furthermore, this analysis is only necessary in the case where $cols(P_{\Delta K_1 \perp \Delta K_2}) = rows(\Psi_{\Delta K_1 \perp \Delta K_2}) = rank((I - U_{\Delta K_1} U_{\Delta K_1}^T) L_{\Delta K_2}) > 0$. If this condition is not met, ΔK_1 and ΔK_2 are orthogonal to each other and X reduces to

$$X = U_X \Sigma_X V_X^T = \begin{bmatrix} U_{\Delta K_1} & U_{\Delta K_2} \end{bmatrix} \begin{bmatrix} \Sigma_{\Delta K_1} & 0 \\ 0 & \Sigma_{\Delta K_2} \end{bmatrix} \begin{bmatrix} V_{\Delta K_1} & V_{\Delta K_2} \end{bmatrix}^T. \tag{18}$$

In order to provide the non-linear fit operation in Parker's method with an appropriate search direction, the Jacobian matrix of the change in the transfer function with respect to $\alpha_{\Delta K_i}$ must also be computed. To compute the Jacobian matrix, we let

$$\Phi = (s^2 M + sG + K), \tag{19}$$

$$\mathrm{H} = \left[\Sigma_X^{-1} + V_X^T \Phi^{-1} U_X\right], \tag{20}$$

$$\frac{\partial \mathrm{H}}{\partial \alpha_{\Delta K_i}} = \left[-\Sigma_X^{-1} \frac{\partial \Sigma_X}{\partial \alpha_{\Delta K_i}} \Sigma_X^{-1} + \frac{\partial V_X^T}{\partial \alpha_{\Delta K_i}} \Phi^{-1} U_X + V_X^T \Phi^{-1} \frac{\partial U_X}{\partial \alpha_{\Delta K_i}} \right], \tag{21}$$

$$\frac{\partial \mathrm{H}^{-1}}{\partial \alpha_{\Delta K_i}} = -\mathrm{H}^{-1} \frac{\partial \mathrm{H}}{\partial \alpha_{\Delta K_i}} \mathrm{H}^{-1}. \tag{22}$$

The Jacobian matrix of the change in the transfer function with respect to $\alpha_{\Delta K_i}$ becomes

$$\frac{\partial \Delta T(s)}{\partial \alpha_{\Delta K_i}} = -s^2 C_a \Phi^{-1} \left[\frac{\partial U_X}{\partial \alpha_{\Delta K_i}} \mathrm{H}^{-1} V_X^T + U_X \frac{\partial \mathrm{H}^{-1}}{\partial \alpha_{\Delta K_i}} V_X^T + U_X \mathrm{H}^{-1} \frac{\partial V_X^T}{\partial \alpha_{\Delta K_i}} \right] \Phi^{-1} F. \tag{23}$$

This, of course, requires the derivation of $\frac{\partial U_X}{\partial \alpha_{\Delta K_i}}$, $\frac{\partial \Sigma_X}{\partial \alpha_{\Delta K_i}}$, and $\frac{\partial V_X}{\partial \alpha_{\Delta K_i}}$. It can be shown using (4), (5), and (6) that

$$\frac{\partial X}{\partial \alpha_{\Delta K_i}} = \Delta K_i = \frac{\partial U_X}{\partial \alpha_{\Delta K_i}} \Sigma_X V_X^T + U_X \frac{\partial \Sigma_X}{\partial \alpha_{\Delta K_i}} V_X^T + U_X \Sigma_X \frac{\partial V_X^T}{\partial \alpha_{\Delta K_i}}. \tag{24}$$

Since U_X is by definition a unitary orthogonal matrix, the following holds

$$U_X^T U_X = I \Rightarrow \frac{\partial U_X^T}{\partial \alpha_{\Delta K_i}} U_X + U_X^T \frac{\partial U_X}{\partial \alpha_{\Delta K_i}} = \Omega_U^T + \Omega_U = 0. \tag{25}$$

Where the anti-symmetric matrix Ω_U is given by

$$\Omega_U = U_X^T \frac{\partial U_X}{\partial \alpha_{\Delta K_i}}. \tag{26}$$

A similar analysis is performed to define the anti-symmetric matrix Ω_V for V_X.

$$\Omega_V = \frac{\partial V_X^T}{\partial \alpha_{\Delta K_i}} V_X. \tag{27}$$

Rearranging (24) by multiplying by U_x^T from the left and V_x from the right and then substituting (26) and (27) we see that

$$\mathrm{Y} = U_X^T \Delta K_i V_X = \Omega_U \Sigma_X + \frac{\partial \Sigma_X}{\partial \alpha_{\Delta K_i}} + \Sigma_X \Omega_V. \tag{28}$$

Due to the anti-symmetric nature of Ω_U and Ω_V, we see that $\frac{\partial \Sigma_X}{\partial \alpha_{\Delta K_i}}$ is the diagonal matrix defined by

$$\left(\frac{\partial \Sigma_X}{\partial \alpha_{\Delta K_i}} \right)_{(j,k)} = \begin{cases} 0, j \neq k \\ y_{(j,k)}, j = k \end{cases}. \tag{29}$$

The elements of Ω_U and Ω_V are calculated by solving the set of linear systems defined by

$$\begin{aligned} \sigma_k \Omega_{U(j,k)} + \sigma_j \Omega_{V(j,k)} &= y_{(j,k)} \\ \sigma_j \Omega_{U(j,k)} + \sigma_k \Omega_{V(j,k)} &= -y_{(k,j)} \end{aligned}, \; j \neq k \tag{30}$$

Since Ω_U and Ω_V are anti-symmetric, computational savings can be gained by only computing the upper or lower triangular regions of these matrices and subtracting the transpose of these triangular matrices from themselves to arrive at Ω_U and Ω_V.

The derivatives $\frac{\partial U_X}{\partial \alpha_{\Delta K_i}}$ and $\frac{\partial V_X}{\partial \alpha_{\Delta K_i}}$ can now be computed by the following

$$\frac{\partial U_X}{\partial \alpha_{\Delta K_i}} = U_X \Omega_U , \tag{31}$$

$$\frac{\partial V_X}{\partial \alpha_{\Delta K_i}} = -V_X \Omega_V . \tag{32}$$

ALGORITHM COMPLEXITY ANALYSIS

From [3], we can assume that the typical upper bound on the execution time of a thin-SVD operation on a $\Re^{m \times m}$ matrix with rank n is $O(mn^2 + n^3)$. Thus, an analysis of the naïve approach where (4) is computed directly from (5) gives us an upper bound on the execution time of $O(N_{dof} n^2 N_{\Delta K}^2 + n^3 N_{\Delta K}^3)$, where n is the number of locations at which damage is estimated and $N_{\Delta K}$ is the rank of individual perturbations ΔK in the stiffness matrix K associated with the monitored damage locations. Assuming that the thin-SVD matrices of all ΔK_i matrices are pre-computed, computation of (4) using low-rank SVD updates in situations where there exists combinations of the ΔK_i matrices that are non-orthogonal gives us an upper bound on the execution time of $O(N_{dof} k^2 N_{\Delta K}^2 + k^3 N_{\Delta K}^3)$, where k is the number of non-orthogonal combinations of locations at which damage is estimated. Observing that *k* is typically less than *n*, it is easy to see that our method is requires less computational complexity than computing (4) directly.

We can extend this analysis to the damage localization technique utilized in [4]. It is evident that using the naïve approach where (4) is computed directly from (5) dictates that the upper bound on the execution time for this operation becomes $O(pr(N_{dof} n^2 N_{\Delta K}^2 + n^3 N_{\Delta K}^3))$, where *p* is the number of possible combinations of damage locations to be monitored and *r* is the maximum number of iterations for each non-linear fit performed. However, for the method proposed in this paper, the upper bound on the execution time for the damage localization operation becomes $O(p_{no} r(N_{dof} k^2 N_{\Delta K}^2 + k^3 N_{\Delta K}^3) + p_o r) = O(p_{no} r(N_{dof} k^2 N_{\Delta K}^2 + k^3 N_{\Delta K}^3))$, where p_{no} is the number of combinations of *k* non-orthogonal damage locations, p_o is the number of combinations of *k* orthogonal damage locations, and *r* is the maximum number of iterations for each non-linear fit performed.

Since the number of non-orthogonal combinations of damage locations is typically much less than the number of all combinations for a particular system. For example, a one-dimensional cantilevered beam finite element model with 36 discrete elements has 630 different possible pairs of damage locations; however, only 35 of these possible pairs consist of non-orthogonal damage locations. This means that for over 94.444% of all combinations, (4) can be constructed as

described in (18). Assuming that the $U_{\Delta K_i}$, $\Sigma_{\Delta K_i}$, and $V_{\Delta K_i}$ matrices are already pre-computed, (18) becomes an $O(1)$ operation – remarkably less than that required for calculating (4) directly from (5). The remaining 5.556% non-orthogonal combinations each require $O(N_{dof} N_{\Delta K}^2 + N_{\Delta K}^3)$ time to execute – still significantly less than that required to compute (4) directly.

SUMMARY

This paper presents a comparatively computationally efficient method (in the face of a naïve approach) for determining multiple locations of damage in a structure. It uses a model of a structure's dynamics and a non-linear optimization technique to estimate changes in the model representative of the frequency response differences between a damaged and undamaged structure. The non-linear optimization technique is coupled with the Sherman-Morrison-Woodbury method for matrix inversion and a low-rank thin-SVD update method for handling non-orthogonal degrees of freedom between damage locations. Like Parker's method, a useful by-product of this method is an estimation of how much change has occurred between the damaged and undamaged model, which corresponds directly with where the damage has occurred on the structure and to what extent.

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